Minimum and Maximum Variance Analysis

BENGT U. Ö. SONNERUP AND MAUREEN SCHEIBLE  
Dartmouth College  
Hanover, NH, U.S.A.

8.1 Introduction

The main purpose of minimum or maximum variance analysis (MVA) is to find, from single-spacecraft data, an estimator for the direction normal to a one-dimensional or approximately one-dimensional current layer, wave front, or other transition layer in a plasma. In the present chapter, we will develop the method in the context of determination of such a unit normal vector, \( \hat{n} \), from minimum variance analysis of magnetic field vector data (MVAB) acquired by a spacecraft as it traverses a current sheet. This is the application where the method was first employed \cite{Sonnerup1967}. Other applications, for example the use of maximum variance analysis of measured electric field vectors (MVAE) for determination of a normal direction, will be discussed briefly toward the end of the chapter. In the context of current-layer traversals by several clustered spacecraft, MVA can be used on data from each individual spacecraft as an initial step. The results can then be used to establish stationarity of current-layer orientation and as a partial benchmark test for more sophisticated analysis tools, e.g., those in Chapters 11 and 15.

The presentation is organised as follows: in Section 8.2, the physical and mathematical basis of MVA is reviewed; in Section 8.3 error estimates are presented; in Section 8.4 other applications are examined, and in Section 8.5 an overview and discussion of results from a test case observed by the spacecraft AMPTE/IRM is provided. Finally, Section 8.6 contains a summary of the main items and precautions that pertain to MVA.

8.2 Theory

8.2.1 Elementary Considerations

The minimum variance analysis technique, applied to magnetic field vector data measured during a spacecraft traversal of a transition layer, is based on an idealised one-dimensional (1-D: \( \partial/\partial x = 0, \partial/\partial y = 0 \)) model of the layer so that only one of the three terms remains in the cartesian expression for the divergence of \( B \):

\[
\nabla \cdot B = \frac{\partial B_z}{\partial z} = 0
\]

In other words, \( B_z \) is independent of \( z \). Here \((x, y, z)\) is a local cartesian coordinate system—unknown \textit{a priori}—with its \( z \) axis pointing along the sought-after vector, \( \hat{n} \), nor-
normal to the layer. It follows from Faraday’s law, $\nabla \times E = -\partial B / \partial t$, that the field component $B_z$ must also be time independent, $\partial B_z / \partial t = 0$, in such an idealised structure so that a spacecraft traversing it would observe a strictly constant value of $B_z$. In that case, only three distinct vector measurements, $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$, are needed to determine $n$. Typically, $B^{(1)}$ and $B^{(3)}$ would be measured on opposite sides of the layer, and $B^{(2)}$ would be measured somewhere near the middle of the layer. Since

$$B^{(1)} \cdot \hat{n} = B^{(2)} \cdot \hat{n} = B^{(3)} \cdot \hat{n}$$  \hspace{1cm} (8.2)

the vectors $(B^{(1)} - B^{(2)})$ and $(B^{(2)} - B^{(3)})$ are tangential to the layer so that their cross product, assuming it is not zero, is along $\hat{n}$:

$$\hat{n} = \pm \frac{(B^{(1)} - B^{(2)}) \times (B^{(2)} - B^{(3)})}{|(B^{(1)} - B^{(2)}) \times (B^{(2)} - B^{(3)})|}$$  \hspace{1cm} (8.3)

Several features of this simple example are of interest:

1. The calculation is not based on the assumption $\mathbf{B} \cdot \hat{n} = 0$, which is sometimes made, but it allows determination of the actual value of the normal field component

$$B_n = \mathbf{B} \cdot \hat{n} = \pm \frac{B^{(1)} \cdot (B^{(2)} \times B^{(3)})}{|(B^{(1)} - B^{(2)}) \times (B^{(2)} - B^{(3)})|}$$

2. Exactly three vectors are needed to obtain a unique determination of $\hat{n}$ and $B_n$, provided the difference vectors $(B^{(1)} - B^{(2)})$ and $(B^{(2)} - B^{(3)})$ in equation 8.3 are not aligned. This means that the three vector components tangential to the current layer cannot be arranged as shown in Figure 8.1a but must be as shown in Figure 8.1b. In the former case, the line ABC itself lies in the tangent plane of the layer but any vector $\hat{n}$ perpendicular to ABC satisfies equation 8.2. In such a situation an additional condition, e.g., $\mathbf{B} \cdot \hat{n} = 0$, is needed in order to obtain a unique $\hat{n}$ vector (e.g., $\hat{n} \propto \pm B^{(1)} \times B^{(3)}$). Note that in Figure 8.1a the electric current in the layer is unidirectional and perpendicular to the line ABC. In Figure 8.1b, the current vectors in the layer, which are perpendicular to the difference vectors, are not unidirectional and a unique (except for sign) $\hat{n}$ vector is obtained from equation 8.3.

3. If the difference vectors are small, i.e., if two or all three of the measured vectors are nearly the same, then equation 8.3 approaches the form zero over zero so that no reliable normal vector is obtained. In practice, this situation is avoided, to the maximum extent that it can, by using one measured vector on each side of the layer and one somewhere near its centre, as mentioned above.

4. If only two (non-aligned) vectors, $B^{(1)}$ and $B^{(3)}$ say, are measured then an additional assumption, such as $\mathbf{B} \cdot \hat{n} = 0$ for tangential discontinuities or coplanarity for shocks (see Chapter 10), is needed in order to obtain a normal vector. But it is noted that $(B^{(1)} - B^{(3)})$ still provides a vector tangential to the layer.

5. If more than three vectors are measured and the current is not unidirectional, then more than one $\hat{n}$ vector determination can be made and, except in ideal circumstances, the resulting vectors are not exactly the same. This is the case to be addressed below.
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Figure 8.1: Projection onto the magnetosheath tangent $xy$ plane of three $B$ vectors measured during spacecraft traversal of a 1-D current sheet. Field $B^{(1)}$ is measured on one side, $B^{(3)}$ on the other side, and $B^{(2)}$ somewhere in the middle of the sheet. Each field vector has the same component $B_n$ pointing along the normal vector, i.e., pointing into the paper. Difference vectors such as $(B^{(1)} - B^{(2)})$ and $(B^{(2)} - B^{(3)})$ are therefore tangential. (a) Difference vectors are colinear so that equation 8.3 fails to yield a normal vector, $\hat{n}$; electric current is unidirectional and perpendicular to line ABC. (b) Difference vectors are not colinear, current is not unidirectional, and equation 8.3 yields a unique $\hat{n}$ vector. The eigenvector triad $(x_1, x_2, x_3)$, where $x_3 = \hat{n}$, is shown.

6. A constant off-set vector in the measured vectors $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$ does not influence the calculated normal direction. However, it influences the value of $B_n$, except in the special case where the off-set vector lies in the tangent plane.

8.2.2 Derivation of Minimum Variance Analysis on Magnetic Field (MVAB)

For real transition layers observed in space there are usually more or less pronounced deviations from the ideal 1-D model described in the previous section. The layer is likely to have 2-D or 3-D internal structures which evolve in time and to have temporal fluctuations in the orientation of its normal as well. In some cases, a systematic temporal change in the normal direction may occur during the spacecraft traversal time. To these effects must be added random as well as systematic measurement errors. For modern magnetometers, the former are usually negligible compared to other uncertainties but the latter can sometimes
arise in the form of a zero-level offset of the magnetometer measuring the field component along the spacecraft spin axis. As pointed out already, a constant offset of this type does not influence the determination of $\hat{n}$ from equation 8.3 and this property will be seen to apply to MVAB as well. Another feature of the real situation is that the high time resolution available in many magnetometer experiments allows many vector measurements, $B^{(m)}$ ($m = 1, 2, 3 \ldots M$), to be made during a traversal.

The minimum variance technique is designed to deal with the situation where some or all of the non-ideal effects mentioned above, except for a systematic temporal change in the normal direction, $\hat{n}$, are present. As the estimate of $\hat{n}$, the method identifies that direction in space along which the field-component set $\{B^{(m)} \cdot \hat{n}\} (m = 1, 2, 3 \ldots M)$ has minimum variance. In other words, $\hat{n}$ is determined by minimisation of

$$\sigma^2 = \frac{1}{M} \sum_{m=1}^{M} \left| \left( B^{(m)} - \langle B \rangle \right) \cdot \hat{n} \right|^2$$

(8.4)

where the average $\langle B \rangle$ is defined by

$$\langle B \rangle \equiv \frac{1}{M} \sum_{m=1}^{M} B^{(m)}$$

(8.5)

and where the minimisation is subject to the normalisation constraint $|\hat{n}|^2 = 1$. Using a Lagrange multiplier, $\lambda$, to implement this constraint, one then seeks the solution of the set of three homogeneous linear equations

$$\frac{\partial}{\partial n_X} \left( \sigma^2 - \lambda (|\hat{n}|^2 - 1) \right) = 0$$

$$\frac{\partial}{\partial n_Y} \left( \sigma^2 - \lambda (|\hat{n}|^2 - 1) \right) = 0$$

$$\frac{\partial}{\partial n_Z} \left( \sigma^2 - \lambda (|\hat{n}|^2 - 1) \right) = 0$$

(8.6)

where $\sigma^2$ is given by equation 8.4 and $\hat{n}$ is represented in terms of its three components $(n_X, n_Y, n_Z)$ along the cartesian coordinate system $X, Y, Z$ (e.g., GSE or GSM) in which the field data $\{B^{(m)}\}$ are given. When the differentiations in equation 8.6 have been performed, the resulting set of three equations can be written in matrix form as

$$\sum_{\nu=1}^{3} M_{\mu \nu}^{B} n_{\nu} = \lambda n_{\mu}$$

(8.7)

where the subscripts $\mu, \nu = 1, 2, 3$ denote cartesian components along the $X, Y, Z$ system and

$$M_{\mu \nu}^{B} \equiv \langle B_{\mu} B_{\nu} \rangle - \langle B_{\mu} \rangle \langle B_{\nu} \rangle$$

(8.8)

is the magnetic variance matrix. It is seen from equation 8.7 that the allowed $\lambda$ values are the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (given here in order of decreasing magnitude) of $M_{\mu \nu}^{B}$. Since $M_{\mu \nu}^{B}$ is symmetric, the eigenvalues are all real and the corresponding eigenvectors, $x_1$, $x_2$, and $x_3$, are orthogonal. The three eigenvectors represent the directions of maximum,
intermediate, and minimum variance of the field component along each vector. Note that
the sense and magnitude of the eigenvectors remain arbitrary so that, for example, \( x_i, kx_i, -x_i, \) and \(-kx_i\) \((i = 1, 2, 3)\) all are valid eigenvectors. The corresponding \( \lambda \) values
represent the actual variances in those field components and are therefore non-negative.
This point becomes clear by writing the matrix \( M_{\mu \nu}^B \) in the eigenvector basis \((x_1, x_2, x_3)\)
where it is diagonal with diagonal terms given by

\[
M_{ii}^B = \langle B_i B_i \rangle - \langle B_i \rangle \langle B_i \rangle = \lambda_i
\]

In summary, the minimum variance analysis consists of constructing the matrix \( M_{\mu \nu}^B \),
defined by equation 8.8 in terms of the measured field data and the cartesian coordinate
system in which the measured data are represented, and then finding the three eigenvalues
\( \lambda_i \), and corresponding eigenvectors \( x_i \), of the matrix. The eigenvector \( x_3 \) corresponding
to the smallest eigenvalue, \( \lambda_3 \), is used as the estimator for the vector normal to the current
sheet and \( \lambda_3 \) itself represents the variance of the magnetic field component along the esti-
mated normal. The eigenvectors \( x_1 \) and \( x_2 \), corresponding to maximum and intermediate
variance, are then tangential to the transition layer and the set \( \{x_1, x_2, x_3\} \) arranged as
a right-handed orthonormal triad provides suitable basis vectors for the local coordinates
\((x, y, z)\) discussed in connection with equation 8.1. More generally, for any measured
set of vectors \( \{B^{(m)}\} \), not necessarily obtained from a spacecraft traversal of a transition
layer or wave front, the eigenvector set of the variance matrix \( M_{\mu \nu}^B \) derived from the data
provides a convenient natural coordinate system in which to display and analyse the data.
Note also that the matrix \( M_{\mu \nu}^B \) is independent of the temporal order of the measured vec-
tors.

If the data set used for the minimum variance calculation has its minimal size, i.e., if
it consists of exactly three vectors, as in Figure 8.1, then one finds \( \lambda_3 = 0 \). For the case
in Figure 8.1a one would also find \( \lambda_2 = 0, \lambda_1 \neq 0 \), and \( x_1 \) would be parallel to ABC. In
this case the variance matrix \( M_{\mu \nu}^B \) is said to be degenerate and all that can be said about
\( x_2 \) and \( x_3 \) is that they are perpendicular to ABC. This situation is discussed in detail in
Section 8.2.5. In Figure 8.1b, one would find \( \lambda_2 \neq 0, \lambda_1 \neq 0 \) and the triad \((x_1, x_2, x_3)\)
would be oriented as shown qualitatively in the figure.

### 8.2.3 Hodogram Representation

The magnetic hodograph is a curve in space constructed by drawing vectors from the
origin, the lengths and directions of which represent the members of the measured set
\( \{B^{(m)}\} \), and then connecting the arrowheads of those vectors by line segments, following
the time sequence in which they were measured. It has become common practice to display
the hodograph in two projections, called magnetic hodograms. The projection onto a plane
tangential to the layer is a plot of the component \( B_1 = B^{(m)} \cdot x_1 \) versus \( B_2 = B^{(m)} \cdot x_2 \)
and the side view projection is a plot of \( B_1 \) versus \( B_3 = B^{(m)} \cdot x_3 \), i.e., versus the normal field
component. Such a hodogram pair is shown in Figure 8.2 for a particular AMPTE/IRM
magnetopause crossing that will serve as our test case in this chapter and also in Chapter 9.
The data given below the plots consist of the eigenvalues \( (\lambda_1, \lambda_2, \lambda_3) \), the eigenvector
components \( (x_{1X}, x_{1Y}, x_{1Z}) \), \( i = 1, 2, 3 \), along the original (GSE) coordinate system, and
the average field components \( (\langle B_1 \rangle, \langle B_2 \rangle, \langle B_3 \rangle) \) along \((x_1, x_2, x_3)\). It is seen that the
normal field component for this event fluctuates with standard deviation \( \sqrt{\lambda_3} = \sqrt{7.08} = \)
Figure 8.2: Hodogram pair for outbound magnetopause traversal by AMPTE/IRM on October 19, 1984, 05:18:20–05:19:26 UT. Units on axes are nT and high-resolution data have been averaged over spacecraft spin period (∼4.35 s). Eigenvalues $\lambda_i (\text{nT})^2$, eigenvectors $x_i (\text{GSE})$ and average field components $\langle B \rangle \cdot x_i (\text{nT})$ are given in the order of decreasing $\lambda$ value: $x_3$ serves as estimator for $\hat{n}$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\lambda_i (\text{nT})^2$</th>
<th>$x_i$</th>
<th>$\langle B \rangle \cdot x_i (\text{nT})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1012.86</td>
<td>(-0.4061, -0.6845, 0.6055)</td>
<td>15.36</td>
</tr>
<tr>
<td>2</td>
<td>138.01</td>
<td>(-0.2886, -0.5326, -0.7957)</td>
<td>-23.40</td>
</tr>
<tr>
<td>3</td>
<td>7.08</td>
<td>(0.8671, -0.4978, 0.0187)</td>
<td>-0.58</td>
</tr>
</tbody>
</table>

2.66 nT around an average value that is near zero ($\langle B_3 \rangle = -0.58 \text{nT}$) so that the current layer appears to be a tangential discontinuity (TD), albeit one having substantial 2-D or 3-D internal substructures which produce the fluctuations in the normal component. These substructures have been analysed in detail by Sonnerup and Guo [1996]. As discussed in Chapter 9, there are also other reasons to believe that our test case is indeed a TD.

Illustrations of the convenience of using of the eigenvector basis for analytical purposes are found in Sections 8.2.4, 8.2.6, and 8.3.1.

In the application to the dayside magnetopause given here, we have chosen $x_1$ to correspond to the maximum variance eigenvector and $x_3$ to the minimum variance eigenvector, directed northward and outward from the Earth, respectively. This choice is convenient because the ordered set $\{x_1, x_2, x_3\}$ is then similar to the boundary normal coordinate axes, ordered as $(L, M, N)$, which were introduced by Russell and Elphic [1979]. However, the choice is not unique: if $x_1$ and $x_3$ are interchanged and if $x_2 \rightarrow -x_2$, the resulting eigenvector basis corresponds qualitatively to the GSE or GSM systems $(X, Y, Z)$ for...
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Figure 8.3: The variance ellipsoid. Eigenvectors, $x_1$, $x_2$, and $x_3$, are shown relative to the system $(X, Y, Z)$ (e.g., GSE) in which the magnetic data are given. The normal direction is estimated to be along $x_3$.

magnetopause crossings in the subsolar region.

8.2.4 Variance Ellipsoid

The variance, $\sigma^2$, of the magnetic field component along an arbitrarily chosen direction, defined by the unit vector $\hat{k}$, say, can be written in terms of the variance matrix as

$$\sigma^2 = \sum_{\mu \nu} k_\mu M^B_{\mu \nu} k_\nu$$  \hspace{1cm} (8.10)

This result follows from equation 8.4 with $\hat{n}$ replaced by $\hat{k}$. If the expression 8.10 is transformed to the eigenvector basis, it becomes

$$\sigma^2 = \lambda_1 k_1^2 + \lambda_2 k_2^2 + \lambda_3 k_3^2$$  \hspace{1cm} (8.11)

where $k_i = \hat{k} \cdot x_i$, $i = 1, 2, 3$. Equation 8.11 invites the definition of a “variance space” in which the coordinates along the eigenvectors $x_1$, $x_2$, and $x_3$ are $\sigma_i = \sqrt{\lambda_i} k_i$, $i = 1, 2, 3$, and in which the variance, $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$, is the distance from the origin. In terms of these coordinates the normalisation condition, $|\hat{k}|^2 = k_1^2 + k_2^2 + k_3^2 = 1$, becomes

$$\frac{\sigma_1^2}{\lambda_1} + \frac{\sigma_2^2}{\lambda_2} + \frac{\sigma_3^2}{\lambda_3} = 1$$  \hspace{1cm} (8.12)

This expression defines the variance ellipsoid, as shown in Figure 8.3, the principal-axes half lengths of which are $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$, and $\sqrt{\lambda_3}$. The distance along an arbitrarily chosen direction in variance space, from the origin to the intersection of a radial line along that direction with the ellipsoid surface, represents the standard deviation of the magnetic field.
component along the chosen direction. However, note that because the transformation from physical space to variance space involves unequal stretching along the three axes ($\sigma_i = \sqrt{\lambda_i} k_i$), directions in the two spaces do not agree, except for the principal directions. If $\hat{k}$ is chosen along one of the principal axes, $\hat{k} = x_i$, then $\sigma^2 = \lambda_i$ as expected.

8.2.5 Degeneracy

When the three eigenvalues of the variance matrix are distinct, the matrix and the variance ellipsoid it represents are said to be non-degenerate. This is the most common situation in practice but there is also a significant number of cases where near degeneracy occurs. Three types of degeneracy are possible: $\lambda_1 \simeq \lambda_2; \lambda_2 \simeq \lambda_3; \text{ and } \lambda_1 \simeq \lambda_2 \simeq \lambda_3$.

The first of these cases, $\lambda_1 \simeq \lambda_2$, corresponds to a discus-shaped (oblate) variance ellipsoid; the minimum variance direction, $x_3$, which is along the axis of the discus, remains well determined but any pair of vectors perpendicular to $x_3$, i.e., any vectors lying in the equatorial plane of the discus, may serve as $x_1$ and $x_2$. This degeneracy, therefore, does not limit the utility of MVAB for normal-vector and normal-field-component determinations, provided $\lambda_3 \ll \lambda_2 \simeq \lambda_1$.

The second type of degeneracy, $\lambda_2 \simeq \lambda_3$, corresponds to a cigar-shaped (prolate) variance ellipsoid with the axis of the cigar along $x_1$ and with the intermediate and minimum variance directions, $x_2$ and $x_3$, being constrained to be perpendicular to $x_1$ but otherwise arbitrary. In this case, no valid direction normal to the layer is obtained from MVAB although $x_1$ remains a good vector tangential to the layer, provided $\lambda_1 \gg \lambda_2 \simeq \lambda_3$. Near degeneracy of this type has been found to be rather common in analyses of magnetopause and geotail current layer crossings. A physical situation where it will occur is when the electric current in the layer is unidirectional or nearly unidirectional so that not only the normal ($B_1 B_3$) hodogram, but also the tangential ($B_1 B_2$) hodogram is a vertical line ($B_2 = \text{const.}$) or approximately a vertical line, i.e., when $\lambda_2 \simeq \lambda_3 \simeq 0$. This situation is the generalisation to many measured vectors of the degenerate case of three measured vectors with a common tangential component, shown in Figure 8.1a, where equation 8.2 fails to provide a normal vector. By contrast, Figure 8.1b corresponds to the non-degenerate case where $\lambda_1 > \lambda_2 > \lambda_3 = 0$ (or in an exceptional case to $\lambda_1 = \lambda_2 > \lambda_3 = 0$). In practice, near degeneracy with $\lambda_2 \simeq \lambda_3 \neq 0$ occurs more frequently than with $\lambda_2 \simeq \lambda_3 \simeq 0$.

The third case of degeneracy $\lambda_1 \simeq \lambda_2 \simeq \lambda_3$ corresponds to a spherical or nearly spherical variance ellipsoid. In this case no information about directions normal or tangential to a layer is obtained. If $\lambda_1 \simeq \lambda_2 \simeq \lambda_3 \simeq 0$, all measured magnetic field vectors are equal while for $\lambda_1 \simeq \lambda_2 \simeq \lambda_3 \neq 0$ the measured field consists of a uniform component (which could be zero) plus magnetic fluctuations having isotropic variance.

When two eigenvalues, $\lambda_2$ and $\lambda_3$, say, are nearly the same, the uncertainty in the corresponding eigenvectors is large with respect to rotation about the remaining eigenvector, $x_1$. In extreme cases, the orientation of the intermediate and minimum variance directions may trade places, corresponding approximately to a $90^\circ$ rotation of $x_2$ and $x_3$ around $x_1$, in response to a minor adjustment of the number of data points or the data filtering used in the analysis. Such behaviour seriously or completely compromises the use of $x_3$ as a predictor of the direction normal to the layer. A case in point is a layer of nearly unidirectional current which contains a string of tearing mode islands. In this situation both $\lambda_2$ and $\lambda_3$ are small but non-zero. Depending on where, and at what angle, the spacecraft trajectory intersects the layer and on how the data interval used for the MVAB is selected,
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Figure 8.4: Eigenvector flip. Eigenvectors $x_2$ and $x_3$ are interchanged, approximately, when 1 data point is deleted (in right hodogram pair) at each end of the original data interval (shown in left hodogram pair). Data are from magnetopause traversal by AMPTE/IRM on September 25, 1984, 05:58:21–05:59:44 UT.

the minimum variance direction, $x_3$, may lie either along the normal to the layer or along the average current vector which is tangent to the layer. An example of actual eigenvector flipping for a noisy magnetopause traversal is shown in Figure 8.4. We see that the eigenvalue ratio in part (a) of the figure is substantial: $\lambda_2/\lambda_3 = 2.05$. Nevertheless, removal of only one point at each end of the data interval leads to a flip and to the result in part (b).

In summary, near degeneracy should result in large error estimates for the corresponding eigenvectors and for the field components along those eigenvectors. Quantitative error estimates having this property will be provided in Section 8.3.

8.2.6 Constraint $\langle B_3 \rangle = 0$

In some circumstances, information may be available from other measurements or from theory to indicate that the current layer should be a tangential discontinuity, i.e., that the average field component, $\langle B \rangle \cdot \hat{n}$, along its normal should be zero. In particular for the degenerate case $\lambda_2 \simeq \lambda_3$, it may be necessary to implement a constraint of this type in order to obtain a useful normal vector prediction. The method often used is to calculate the average of a set of field vectors measured on one side of the layer and the average of another set measured on the opposite side and then use the cross product of the resulting two vectors as the predictor of the normal vector, $\hat{n}$. However, there is an alternate approach [Sonnerup and Cahill, 1968] in which all field vectors measured within the layer...
as well as vectors measured on its two sides are utilised: $\hat{n}$ is chosen such that the variance of the field component along it is a minimum, subject to the constraint $\langle B \rangle \cdot \hat{n} = 0$. Geometrically, this condition requires $\hat{n}$ to lie in a plane perpendicular to $\langle B \rangle$ (with the directions of $\hat{n}$ and $\langle B \rangle$ converted to corresponding directions in variance space). The intersection of this plane with the variance ellipsoid is an ellipse, as shown in Figure 8.5, the minor axis of which is used as the predictor for $\hat{n}$.

The quantitative analysis is carried out most conveniently by minimising the expression $\left[ \sigma^2 - \lambda (|\hat{n}|^2 - 1) - 2 \Gamma \langle B \rangle \cdot \hat{n} \right]$ where $\lambda$ and $2 \Gamma$ are Lagrange multipliers used to implement the constraints $|\hat{n}|^2 = 1$ and $\langle B \rangle \cdot \hat{n} = 0$, respectively. Partial differentiation with respect to each of the three components of $\hat{n}$ then leads to the following set of three linear non-homogeneous equations

$$\sum_{\nu=1}^{3} M^{B}_{\mu \nu} \cdot n_{\nu} - \lambda n_{\mu} = \Gamma \langle B_{\mu} \rangle \quad \mu = 1, 2, 3 \quad (8.13)$$

where $M^{B}_{\mu \nu}$ is given by equation 8.8. Using the eigenvectors of $M^{B}_{\mu \nu}$ as basis vectors and assuming $\lambda \neq \lambda_i$, we find

$$n_i = \Gamma \langle B_i \rangle / (\lambda_i - \lambda) \quad i = 1, 2, 3 \quad (8.14)$$

where $(n_1, n_2, n_3)$ and $(\langle B_1 \rangle, \langle B_2 \rangle, \langle B_3 \rangle)$ are the components of $\hat{n}$ and $\langle B \rangle$ along the maximum, intermediate, and minimum variance eigenvectors, respectively. The normalisation
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constraint, $|\hat{n}|^2 = 1$, then gives

$$\Gamma = \pm \left[ \sum_{i=1}^{3} (B_i)^2 / (\lambda_i - \lambda) \right]^{-1/2}$$

(8.15)

and, using equation 8.14, the constraint $\langle \mathbf{B} \rangle \cdot \hat{n} = 0$ gives

$$\sum_{i=1}^{3} (B_i)^2 / (\lambda_i - \lambda) = 0$$

(8.16)

which is a quadratic equation for the Lagrange multiplier $\lambda$. The two roots of this equation are denoted by $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$. The major and minor semiaxes of the ellipse of intersection shown in Figure 8.5 are $\sqrt{\lambda_{\text{max}}}$ and $\sqrt{\lambda_{\text{min}}}$, respectively, so that $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$ represent the maximum and minimum values of the variance of the set $\{\mathbf{B}^{(m)} \cdot \hat{n}\}$ when the constraint $\langle \mathbf{B} \rangle \cdot \hat{n} = 0$ has been implemented. The sought-after normal vector, $\hat{n}$, is obtained by substituting $\lambda_{\text{min}}$ into equations 8.15 and 8.14; the desired sense of $\hat{n}$ is obtained by properly selecting the sign of $\Gamma$ in equation 8.15.

In a more general formulation [A. V. Khrabrov, private communication], one can show that a constraint of the form $\hat{n} \cdot \hat{e} = 0$, where $\hat{e}$ is a known unit vector, leads to the eigenvalue problem $\mathbf{P} \cdot \mathbf{M}^{\mathbf{B}} \cdot \mathbf{P} \cdot \hat{n} = \lambda \hat{n}$. Here $\mathbf{P}$ is the matrix describing projection of a vector onto the plane perpendicular to $\hat{e}$, i.e., $P_{ij} = \delta_{ij} - e_i e_j$. By putting $\hat{n} = \hat{e}$ in the eigenvalue equation, it is then seen that $\hat{e}$ is an eigenvector corresponding to $\lambda = 0$. In the application above, the other two eigenvalues are the same as $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ from equation 8.16 with their corresponding eigenvectors being the same as those from equation 8.14.

An alternative approach to the problem of finding the vector normal to a tangential discontinuity was described by Siscoe et al. [1968] who minimised the sum of the squares of the individual normal field components, subject to the constraint $|\hat{n}|^2 = 1$. In other words, they extremised

$$\frac{1}{M} \sum_{m=1}^{M} (\mathbf{B}^{(m)} \cdot \hat{n})^2 - \lambda \left( |\hat{n}|^2 - 1 \right)$$

(8.17)

which leads to the problem of finding the eigenvalues and eigenvectors of the matrix

$$\tilde{\mathbf{M}}^B_{\mu \nu} = \langle B_\mu B_\nu \rangle$$

Table 8.1 contains a comparison, for the AMPTE/IRM event in Figure 8.2, of normal vectors and normal field components obtained (i) from minimum variance analysis without constraint; (ii) from minimum variance analysis with constraint $\langle \mathbf{B} \rangle \cdot \hat{n} = 0$; and (iii) from minimisation of $\langle (\mathbf{B} \cdot \hat{n})^2 \rangle$. It is seen that, in this particular case where the actual normal field component was very small or zero, all three calculations give very similar results, the maximum angular deviation being less than $1.4^\circ$. In cases where the actual normal-field component is substantial, the results from the three methods differ significantly and only method (i) is appropriate.

The method (iii) developed by Siscoe et al. has an alternate important application that will be illustrated later on. It can be used to characterise a set of nearly aligned normal-vector estimates in terms of their average direction (Siscoe et al. eigenvector corresponding to the maximum eigenvalue) and the elliptical cross-section of their error cone (Siscoe et al. eigenvectors and square roots of intermediate and minimum eigenvalues).
Table 8.1: Predictions of normal vector $\mathbf{\hat{n}}$ for magnetopause of tangential-discontinuity type observed by AMPTE/IRM on October 19, 1984 (05:18:20–05:19:26 UT).

<table>
<thead>
<tr>
<th>Method</th>
<th>$n_x$</th>
<th>$n_y$</th>
<th>$n_z$</th>
<th>$\langle \mathbf{B} \cdot \mathbf{\hat{n}} \rangle$ (nT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i): Min. variance (MVAB)</td>
<td>0.8671</td>
<td>-0.4978</td>
<td>0.0187</td>
<td>-0.58</td>
</tr>
<tr>
<td>(ii): MVAB + $\langle \mathbf{B} \cdot \mathbf{\hat{n}} \rangle \equiv 0$</td>
<td>0.8728</td>
<td>-0.4865</td>
<td>0.0386</td>
<td>0.00</td>
</tr>
<tr>
<td>(iii): Min. $\langle \mathbf{B} \cdot \mathbf{\hat{n}} \rangle^2$</td>
<td>0.8718</td>
<td>-0.4887</td>
<td>0.0349</td>
<td>-0.11</td>
</tr>
</tbody>
</table>

8.3 Error Estimates

Uncertainties in the orientations of the eigenvectors $(x_1, x_2, x_3)$ and in the values of the average field components $\langle \mathbf{B} \cdot \mathbf{x}_i \rangle$ can be either purely statistical in nature or can be caused by lack of stationarity or quasi-one-dimensionality of the structure being studied. The statistical errors can be estimated by use of standard analytical tools or by use of computational techniques such as the bootstrap method. These approaches are illustrated and discussed below. A comparison of normal vectors and of errors is provided in Section 8.5. Uncertainties associated with lack of stationarity or other systematic effects are more difficult to assess but can be addressed, at least in part, by use of nested data segments: this approach is also presented.

8.3.1 Analytical Estimates of Statistical Errors

As mentioned already, it is difficult to estimate the actual error in the normal vector obtained from MVA because, in reality, the error may have a systematic part in addition to a part caused by finite sampling of a stationary noise component in the measured field vectors $\mathbf{B}^{(m)}$. In what follows we assume that no systematic errors are present: we are therefore concerned only with the noise component for which it is expected that the error should decrease as $M^{-1/2}$ as the number of measured field vectors, $M$, used in the analysis, increases. Following a recent development by Khrabrov and Sonnerup [1998a] (which was motivated by the preparation of this chapter), the uncertainties in the directions of the eigenvectors $(x_1, x_2, x_3)$ of the variance matrix, $\mathbf{M}^B \equiv \mathbf{M}$ (the superscript $B$ is suppressed for brevity), calculated from the measured field vectors, are estimated by performing perturbation analysis on the eigenvector equation 8.7 around the unknown noise-free state which is denoted by an asterisk:

$$(\mathbf{M}^* + \Delta \mathbf{M}) \cdot (x_i^* + \Delta x_i) = (\lambda_i^* + \Delta \lambda_i)(x_i^* + \Delta x_i)$$

Equation 8.18 is now written in the unperturbed eigenbasis in which $\mathbf{M}^*$ is diagonal. Using subscript notation (but not the summation convention), the $j$th component of equation 8.18 becomes after simple rearrangements,

$$(\lambda_j^* - \lambda_i^*) \Delta x_{ij} = -\Delta M_{ij} - \Delta \lambda_i \delta_{ij}$$
where $\Delta x_{ij}$ is the $j^{th}$ component of the vector $\Delta x_i$ and where use has been made of $x^*_i = \delta_{ij}$. Since $M$ and $M^*$, are both symmetric matrices, we have $\Delta M_{ij} = \Delta M_{ji}$. From equation 8.19, one then concludes that $\Delta x_{ij} = -\Delta x_{ji}$ which expresses the fact that the perturbed eigenvectors must form an orthonormal triad. The perturbed eigenvectors must also retain unit length so that in the linear approximation $\Delta x_{ii} = x^*_i \cdot \Delta x_i = 0$. For $j = i$, equation 8.19 gives $\Delta \lambda_i = \Delta M_{ii}$ while the other components ($j \neq i$) give

$$\begin{align*}
\Delta x_{31} &= -\Delta x_{13} = -\Delta M_{13}/(\lambda_1^* - \lambda_3^*) \\
\Delta x_{32} &= -\Delta x_{23} = -\Delta M_{23}/(\lambda_2^* - \lambda_3^*) \\
\Delta x_{21} &= -\Delta x_{12} = -\Delta M_{21}/(\lambda_1^* - \lambda_2^*)
\end{align*}$$

(8.20)

In the linear approximation, the quantities $\Delta x_{31}$ and $\Delta x_{32}$ also represent the angular rotations (in radians) of the eigenvector $x_3$ towards $x_1$ and $x_2$, respectively. Similarly, $\Delta x_{21}$ represents the angular rotation of $x_2$ towards $x_1$. Since the noise-free state represents an ideal one-dimensional current layer of fixed orientation we have $\lambda_3^* = 0$ in equation 8.20.

The next step is to evaluate the ensemble averages, denoted by the double bracket $\langle\langle \ldots \rangle\rangle$, of $(\Delta x_{31})^2$, $(\Delta x_{32})^2$ and $(\Delta x_{21})^2$. As is seen from equation 8.20 these averages are proportional to $\langle\langle (\Delta M_{13}^2) \rangle\rangle$, $\langle\langle (\Delta M_{23}^2) \rangle\rangle$ and $\langle\langle (\Delta M_{21}^2) \rangle\rangle$, respectively. By ensemble average we mean the average over a large number of realisations of the noise component of the measured field. In reality, only one such realisation is available to us, namely the one contained in the measured set of vectors, $B^{(m)}$. Furthermore, for this set we do not know a priori what part of $B^{(m)}$ is noise. Nevertheless, we can obtain a formal expression for $\Delta M_{ij}$ by replacing $B^{(m)}$ by $(B^{(m)} + \Delta B^{(m)})$ in the definition 8.8 of $M$, where $\Delta B^{(m)}$ is the noise component. The noise is assumed to be stationary, isotropic, and spatially uncorrelated (the final error formulas may in fact work under less restrictive conditions) and, as a result, can be shown to have the following properties:

$$\begin{align*}
\langle\langle \Delta B_i^{(m)} \rangle\rangle &= 0 \
\langle\langle \Delta B_i^{(m)} \Delta B_j^{(m)} \rangle\rangle &= \delta_{ij} \delta_{mn} \langle\langle (\Delta B_3^{(m)})^2 \rangle\rangle \\
\langle\langle \Delta B_i^{(m)} \Delta B_j^{(m)} \Delta B_k^{(p)} \rangle\rangle &= 0 \\
\langle\langle \Delta B_i^{(m)} \Delta B_j^{(m)} \Delta B_k^{(p)} \Delta B_l^{(q)} \rangle\rangle &= \left( \delta_{mn} \delta_{kl} + \delta_{ik} \delta_{jq} + \delta_{iq} \delta_{jk} \right) \cdot \langle\langle (\Delta B_3^{(m)})^2 \rangle\rangle^2
\end{align*}$$

(8.21)

In the fourth property, we have used the notation $\delta_{ij} = \delta_{ij} \delta_{mn}$; we have also assumed the noise to be normally distributed. The second property indicates that the variance of each of the three vector components of the noise has been assumed to be the same (isotropy) and that it is represented by the variance of the component along $x_3^*$. Furthermore, the assumption of time stationarity means that $\langle\langle (\Delta B_3^{(m)})^2 \rangle\rangle$ is, in fact, independent of $m$. As shown by Khrabrov and Sonnerup [1998a], it is also equal to $\langle\langle \lambda_3 \rangle\rangle M/(M - 1)$; similarly, they show that $\langle\langle \lambda_2 \rangle\rangle = \lambda_2^* + \langle\langle \lambda_3 \rangle\rangle$ and that $\langle\langle \lambda_1 \rangle\rangle = \lambda_1^* + \langle\langle \lambda_3 \rangle\rangle$. Although the ensemble averages, $\langle\langle \lambda_1 \rangle\rangle$, $\langle\langle \lambda_2 \rangle\rangle$, and $\langle\langle \lambda_3 \rangle\rangle$, deviate somewhat from the corresponding eigenvalues, $\lambda_1$, $\lambda_2$, and $\lambda_3$, calculated from the actually measured field vectors, $B^{(m)}$, we will use the set $\{\lambda_1, \lambda_2, \lambda_3\}$ to replace the ensemble averages in the error estimates. The errors associated with this replacement can be estimated by evaluating $\langle\langle (\Delta \lambda_i)^2 \rangle\rangle = \langle\langle (\Delta M_{ii})^2 \rangle\rangle$ for $i = 1, 2, 3$.

We may finally express the perturbed matrix elements, $\Delta M_{ij}$, in terms of $\Delta B^{(m)}$ by use of the definition of $M$, then square $\Delta M_{ij}$, perform the ensemble average and do the sums.
over \( m \). In this development the properties 8.21 are used. The detailed steps [Khrabrov and Sonnerup, 1998a] are straight-forward but too lengthy to be given here. Neglecting terms of order \( \varepsilon^2 \equiv [\lambda_3/(\lambda_2 - \lambda_3)]^2/(M - 1)^2 \) compared to unity, the result is

\[
\langle\langle (\Delta M_{ij})^2 \rangle\rangle = \langle\langle \lambda_3 \rangle\rangle \left[ \langle\langle \lambda_{ij} \rangle\rangle - \langle\langle \lambda_3 \rangle\rangle \right] (1 + \delta_{ij})/(M - 1) \quad (8.22)
\]

so that the angular error estimates (in radians) become

\[
|\Delta\varphi_{ij}| = |\Delta\varphi_{ji}| = \sqrt{\langle\langle (\Delta x_{ij})^2 \rangle\rangle} = \sqrt{\langle\langle (\Delta x_{ji})^2 \rangle\rangle} = \sqrt{\frac{\lambda_3}{(M - 1)} \frac{(\lambda_i + \lambda_j - \lambda_3)}{(\lambda_i - \lambda_j)^2}}, \quad i \neq j \quad (8.23)
\]

Here \( |\Delta\varphi_{ij}| \) denotes the expected angular uncertainty of eigenvector \( x_i \) for rotation toward or away from eigenvector \( x_j \). It is noted that, except for very small values of \( \lambda_3/(M - 1) \), the uncertainty becomes large for the nearly degenerate case, \( \lambda_i \approx \lambda_j \), discussed in Section 8.2.5. This is an expected and desirable property but it must be remembered that the linear analysis used in producing the error estimate then breaks down. Such cases are nevertheless of interest. For example, if \( \lambda_1 \approx \lambda_2 \) then \( x_3 \) remains a good normal vector provided \( |\Delta\varphi_{31}| \) and \( |\Delta\varphi_{32}| \) are small. And if \( \lambda_3 \approx \lambda_2 \ll \lambda_1 \) then \( x_1 \) remains a good tangent vector to the current sheet provided \( |\Delta\varphi_{21}| \) and \( |\Delta\varphi_{13}| \) are small. An important use of this tangent vector is discussed in Sections 8.4.1 and 8.4.2. Another situation where \( \lambda_2 \approx \lambda_3 \) is likely to occur is in maximum variance analysis of electric field data (MVAE). In this application, the maximum variance direction, \( x_1 \), is normal to the current layer, as discussed further in Section 8.4.2. In reality, the ordering \( \lambda_3 < \lambda_2 \ll \lambda_1 \), or even \( \lambda_3 \ll \lambda_2 \ll \lambda_1 \), is often found in MVAE, indicating a lack of isotropy of the noise (defined as the deviations from a 1-D time-independent structure); in that case the error estimate \( |\Delta\varphi_{12}| \) should be calculated with \( \lambda_2 \) replacing \( \lambda_3 \) in equation 8.23. Additionally it is noted that non-isotropic noise may lead to a bias (a systematic error) in the normal vector estimate.

The statistical uncertainty in the component of the average magnetic field along the eigenvector \( x_3 \) is composed of three parts: the uncertainty in the average associated with the corresponding variance \( \lambda_3 \) and the two uncertainties associated with the angular error estimates for \( x_3 \). Assuming these errors to be independent, we can then write the composite statistical error estimate for \( \langle B \rangle \cdot x_3 \) as

\[
|\Delta\langle B \cdot x_3 \rangle| = \sqrt{\frac{\lambda_3}{M - 1} + (\Delta\varphi_{32}(B) \cdot x_2)^2 + (\Delta\varphi_{31}(B) \cdot x_1)^2} \quad (8.24)
\]

Similar expressions can be written for the uncertainties in \( \langle B \rangle \cdot x_1 \) and in \( \langle B \rangle \cdot x_2 \) but these error estimates are usually of less interest.

As an example, the error estimates for the magnetopause crossing depicted in Figure 8.2 have been calculated from equations 8.23 and 8.24, with the following results:

\( \Delta\varphi_{32} = \Delta\varphi_{23} = \pm 0.062 \text{ rad} \approx \pm 3.6^\circ; \Delta\varphi_{31} = \Delta\varphi_{13} = \pm 0.022 \text{ rad} \approx \pm 1.3^\circ; \Delta\varphi_{12} = \Delta\varphi_{21} = \pm 0.027 \text{ rad} \approx \pm 1.5^\circ; \Delta\langle B \cdot x_3 \rangle = \pm 1.63 \text{ nT} \). A variety of other applications of equations 8.23 and 8.24 will be given later on in this chapter.

### 8.3.2 Bootstrap Error Estimates

It has been proposed recently [Kawano and Higuchi, 1995] that the so-called bootstrap method may provide superior error estimates for the minimum variance analysis in the
8.3. Error Estimates

Figure 8.6: Bootstrap distributions for $N = 1000$ bootstrap sets $\{B^{(m)}(i)\}$ where $m = 1, 2, \ldots, 16$ and $i = 1, 2, \ldots, 1000$: (a) distribution of the normal field component, $(B^{(m)} \cdot x_3)^{(i)}$; (b) distribution of angular deviation (radians) of $x_3^{(i)}$ from vector $x_3$ toward vector $x_1$ (rotation about $x_2$), with $(x_1, x_2, x_3)$ given in Figure 8.2; (c) same for angular deviation of $x_3^{(i)}$ from $x_3$ toward $x_2$ (rotation about $x_1$).
sense that fewer assumptions are needed and that errors associated with time variance, non-planar effects, etc., are automatically incorporated. Although the utility of the method has not been tested extensively with real magnetopause data sets, it seems important to describe the bootstrap procedure briefly.

As described by Kawano and Higuchi [1995], the method is based on performing a very large number of minimum variance calculations, using for each calculation a bootstrap data sample generated from the measured vectors as follows. One vector is drawn at random, with replacement, from the measured set of \( M \) vectors. A second drawing is made in the same fashion and the process is continued until \( M \) drawings have been made. The resulting bootstrap sample thus consists of \( M \) vectors, all of which were measured but some of which may be identical so that a sample usually does not contain all the measured vectors. A large number (\( N \geq 10^5 \), say) of bootstrap samples is generated in this fashion and each sample is subjected to MVA thus producing a set of \( N \) minimum variance eigenvectors \( \{x_3\} \) and corresponding normal field components \( \{B \cdot x_3\} \). The distribution of these quantities can then be characterised in terms of averages, variances, skewness factors, etc. The square root of the variance (the standard deviation) is the quantity that can be directly compared to the error estimates in Section 8.3.1.

As an illustration, we have analysed the 16 vector data samples that form the basis of the minimum variance calculation illustrated in Figure 8.2. A total of \( N = 10^3 \) bootstrap samples were generated from this data set and the MVA was performed on each. The resulting distribution of normal field components and of the angular deviations of the minimum variance eigenvectors \( \{x_3\} \) toward the eigenvectors \( x_1 \) and \( x_2 \), with \( (x_1, x_2, x_3) \) coming from the measured set and given in Figure 8.2, are shown in Figure 8.6 along with the averages and standard deviations. These quantities are found to remain relatively insensitive to \( N \), the number of bootstrap samples, at least for \( N \geq 10^3 \) (rough estimates can be obtained for substantially smaller \( N \) values). The bootstrap averages are close to the single-sample averages in Figure 8.2 and the bootstrap standard deviations for \( \hat{n} \) are about the same as the estimates in Section 8.3.1. It can be shown, based on [Khrabrov and Sonnerup, 1998a], that this agreement is not coincidental. The bootstrap standard deviation for \( \{B \cdot n\} \) (1.17 nT, see Figure 8.6a) is smaller than the value obtained analytically (1.63 nT, see Section 8.3.1). Because questions remain concerning the proper application of the bootstrap technique to MVA, we recommend use of the analytical results for which the underlying assumptions are transparent. Further discussion and application of the bootstrap technique may be found in Chapter 9.

An estimator for the normal vector, \( \hat{n} \), can be obtained as the normalised component-by-component average of the individual bootstrap normals \( \hat{n}^{(i)} \) (\( i = 1, 2, \ldots, 1000 \)). The result is \( \hat{n} = (0.8669, -0.4982, 0.0182) \) which very nearly agrees with the vector \( x_3 \) from Figure 8.2, the angle between the two vectors being only 0.04°. It follows from [Khrabrov and Sonnerup, 1998a] that this agreement is not accidental but is correct including terms of order \( \epsilon \) (but not \( \epsilon^2 \); \( \epsilon \) was defined in connection with equation 8.22). The bootstrap average of the normal field component is -0.60 nT. An alternate estimator for the average normal, \( \langle \hat{n} \rangle \), can be obtained from the \( N \) bootstrap normals by maximising the sum of the squares of the individual components \( \hat{n}^{(i)} \cdot \langle \hat{n} \rangle \). This procedure reduces to the Siscoe et al. least-squares problem (equation 8.17) with \( B^{(m)} \) replaced by \( \hat{n}^{(i)} \) and with the eigenvector of the largest rather than the smallest eigenvalue representing \( \langle \hat{n} \rangle \). For our event, this procedure leads to very nearly the same \( \hat{n} \) vector as above. The square root of
the intermediate eigenvalue and of the smallest eigenvalue from the least-squares problem considered by Siscoe et al. and the corresponding eigenvectors define an uncertainty cone of elliptic cross section for the bootstrap normal vector. The size and orientation of this cone is found to be consistent with the standard deviations shown in Figures 8.6b and 8.6c and also with the results from equation 8.23 (for further discussion, see Section 8.5). It is concluded that the most convenient way to characterise the ensemble of bootstrap normal vectors is to apply the procedure of Siscoe et al. to them.

8.3.3 Other Error Estimates

Three other error estimates for MVAB can be found in the literature and will be mentioned briefly here.

Sonnerup [1971] proposed angular error estimates for \(x_3\) consisting of the angular change produced by assuming the variance ellipsoid to remain unperturbed but changing the minimum variance away from \(\lambda_3\) by the amount \(|\Delta \lambda_3|\) given by

\[
|\Delta \lambda_3|^2 = \left\langle \left(\Delta M_{33} \right)^2 \right\rangle = \frac{1}{(M - 1)M} \sum_{m=1}^{M} \left[ \left( B^{(m)} \cdot x_3 - \langle B \rangle \cdot x_3 \right)^2 - \lambda_3 \right]^2
\]  

(8.25)

This expression for \(|\Delta \lambda_3|^2\) is simply the variance of \(\lambda_3\) divided by \((M - 1)\). From the two conditions

\[
(x_3 + \Delta x_3) \cdot M^B \cdot (x_3 + \Delta x_3) = \lambda_3 + \Delta \lambda_3
\]

\[
(x_3 + \Delta x_3)^2 = 1,
\]

both carried to second order in \(\Delta x_3\), one then finds the expression

\[
|\Delta \lambda_3| = \sum_{j=1}^{3} (\Delta x_{3j})^2 (\lambda_i - \lambda_3)
\]  

(8.27)

The term \(j = 3\) is equal to zero in the sum so that equation 8.27 defines a quadratic relationship between the two components of \(\Delta x_3\) that are perpendicular to \(x_3\). It describes an error cone of elliptical cross section with major axis of half length \(|\Delta \varphi_3| = (|\Delta \lambda_3|/(\lambda_2 - \lambda_3))^{1/2}\) along \(x_2\) and minor axis of half length \(|\Delta x_{31}| = |\Delta \varphi_{31}| = (|\Delta \lambda_3|/\lambda_1 - \lambda_3)^{1/2}\) along \(x_1\). In other words, we can express the principal semiaxes of the cone of uncertainty as

\[
|\Delta \varphi_{3i}| = \sqrt{|\Delta \lambda_3|/(\lambda_i - \lambda_3)}\quad i = 1, 2
\]  

(8.28)

The values \(|\Delta \lambda_{31}|\) and \(|\Delta \lambda_{32}|\) from equation 8.28 can then be used in 8.24 to obtain \(|\Delta \langle B \cdot x_3 \rangle|\). For the magnetopause crossing depicted in Figure 8.2 the resulting error estimates are \(\Delta \lambda_3 = 3.06\) (nT)\(^2\); \(\Delta \varphi_{32} = \pm 8.8^\circ\); \(\Delta \varphi_{31} = \pm 3.2^\circ\); \(\Delta \langle B \cdot x_3 \rangle = \pm 3.74\) nT. It is seen that the uncertainties predicted from this model are more than twice as large as those given in Sections 8.3.1 and 8.3.2.

Kawano and Higuchi [1995] have pointed out that when the field component \(B^{(m)} \cdot x_3\) is normally distributed the expression 8.25 for \(|\Delta \lambda_3|^2\) can be replaced by the simpler formula

\[
|\Delta \lambda_3|^2 = \left\langle \left(\Delta M_{33} \right)^2 \right\rangle = 2\lambda_3^2/(M - 1)
\]  

(8.29)
which can be obtained from equation 8.22 by putting \( i = j = 3 \). For the magnetoopause crossing in Figure 8.2, this expression leads to \( \Delta \lambda_3 = 2.59 \text{ (nT)}^2 \) which should be compared to \( \Delta \lambda_3 = 3.06 \text{ (nT)}^2 \) from equation 8.25. By comparison of the combination of equations 8.28 and 8.29 to the estimate 8.23 with \( j = 3 \) we see that the error obtained from equations 8.28 and 8.29 is larger than that from 8.23 by the factor \( (1 - \lambda_3/\lambda_1)^{1/2} [2(M - 1)]^{1/4} \). While the errors predicted by equation 8.23 exhibit the \( M^{-1/2} \) dependence expected from stationary random noise, those predicted by 8.28 have a weaker \( M^{-1/4} \) dependence. The latter behaviour may perhaps be justified when significant systematic errors (assumed independent of \( M \)) are present in addition to the noise. But for stationary isotropic noise, equation 8.23 has better theoretical justification than 8.28. Accordingly, with the caveats provided in Section 8.3.5, we recommend that, in the future, equation 8.23 be used. However, there is a need to perform benchmark tests of this equation by use of a variety of synthetic data sets.

Hoppe et al. [1981] used qualitative arguments to motivate the following formulas for the angular uncertainties in \( x_3 \):

\[
\Delta \varphi_{31} = \pm \tan^{-1} \left[ \lambda_3 / (\lambda_1 - \lambda_3) \right] \tag{8.30}
\]
\[
\Delta \varphi_{32} = \pm \tan^{-1} \left[ \lambda_3 / (\lambda_2 - \lambda_3) \right]
\]

These formulas do not contain \( M \), the number of data points used in the analysis. Therefore, in contrast to the estimates in Sections 8.3.1 and 8.3.2, these error estimates would not approach zero as \( M \to \infty \), as expected for time-stationary statistical fluctuations. They are directly inconsistent with equation 8.23 or with equations 8.28 and 8.29. For the event in Figure 8.2, they produce the following error estimates: \( \Delta \varphi_{31} = \pm 0.4^\circ \); \( \Delta \varphi_{32} = \pm 3.1^\circ \). It appears that \( \Delta \varphi_{31} \) is unrealistically small. Accordingly, we recommend against the use of equation 8.30.

A third, more extensive study of minimum variance errors was performed by Lepping and Behannon [1980]. It was based on a large number of simulated current sheet crossings in which the unperturbed tangential magnetic field rotated by an angle \( \omega \), but maintained constant magnitude while the normal field component was strictly constant (either non-zero or zero). To this unperturbed configuration was added isotropic noise of different amplitudes. The number of data points used in the MVA was proportional to the angle change \( \omega \), an assumption that is perhaps reasonable for solar-wind discontinuities but not necessarily for other current sheets. They encoded their error estimate for the normal field component, \( \langle B \rangle \cdot x_3 \), in a complicated purely empirical formula that, because of the special nature of the simulated structures, would seem to be of limited general use. If that formula is nevertheless used to estimate the error in \( \langle B \rangle \cdot x_3 \) for the event in Figure 8.2, the result is \( \Delta \langle B \cdot x_3 \rangle \equiv \pm 1.9 \text{ nT} \), which is comparable to the error estimate in Section 8.3.1. However, the reader may wish to take note of Lepping and Behannon’s claim that the formulas 8.25 and 8.28 usually underestimate the errors for the type of discontinuities they studied.

8.3.4 Stationarity

A partial check on time-stationarity and further testing of the error estimates can be obtained by doing MVAB on sets of nested data segments centred at or near the middle of the current sheet. Each larger member of a nest is generated by adding one data point at each end of the preceding segment. The smallest member of a nest has three data points.
8.3. Error Estimates

Figure 8.7: Results of minimum variance analysis for nested data segments starting with $M = 3$ vectors and then increasing to $M = 27$. The nest is centred at 05:18:50 UT. Each vector is the (4.35 s) average of $Q \approx 140$ high-resolution $B$ measurements. Top panel shows normal magnetic field components and error bars for each $M$ value. The quantities $\Delta \phi_{31}$ and $\Delta \phi_{32}$ indicate angular deviations (radians) from the reference normal $x_3$, toward $x_1$ and $x_2$, respectively, where the set $(x_1, x_2, x_3)$ is specified in Figure 8.2. Error bars are calculated from equations 8.23 and 8.24.

The size of the largest member is limited by practical constraints such as the presence of magnetic structures, e.g., unrelated wave modes or turbulence, adjoining the current layer under study. If the normal vector and normal field component are strictly time stationary, then the results from all the different nested segments should be the same. In reality a group of the shortest and longest segments often gives results that are significantly different from those obtained for segments of intermediate duration. Within the intermediate range, the results of MVAB should be the same, or nearly the same, regardless of segment duration if the crossing is to be considered approximately time stationary. In other words, if the results of the MVAB ($\langle B \rangle \cdot x_3$ and the direction of $x_3$) are plotted as functions of the segment length $M$ (the segment duration is $T = (M - 1) \tau$, $\tau$ being the sampling interval), a plateau region should exist for intermediate $M$ values: the wider and flatter the plateau the more nearly does the crossing exhibit time stationarity. The situation is illustrated in Figure 8.7 for the AMPTE/IRM event in Figure 8.2. Also shown are the statistical error bars calculated as described in Section 8.3.1. These error bars define a lower and an upper envelope curve for the expected values of the variable. Note that the error-bar lengths for...
\( \langle B \rangle \cdot x_3 \) and \( \Delta \varphi_{32} \) do not decrease as \( M \) increases, indicating that \( \lambda_3 \) is not constant but increases with \( M \).

This result indicates that the noise is not time stationary. On the other hand, the near constancy of the three variables plotted in Figure 8.7 for \( M \geq 9 \) suggests that approximate time stationarity of the average magnetopause structure is at hand. The plateau average for the normal vector, calculated from the Siscoe et al. least-squares method, is \( \hat{n} = (0.8616, -0.5074, 0.0096) \) and the average field along it is \( \langle B \rangle \cdot \hat{n} = 0.90 \text{nT} \). The vector \( \hat{n} \) forms an angle of 0.97° with the \( x_3 \) vector from Figure 8.2; the error cone associated with the plateau normals will be discussed in Section 8.5.

### 8.3.5 Data Filtering and Optimal Analysis

Current layers observed in space often contain ELF magnetic fluctuations and various short-duration substructures of substantial amplitude. Such fluctuations influence the results of MV AB by increasing \( \lambda_3 \) and sometimes also by introducing a bias (systematic error) in the estimate for the normal vector, \( \hat{n} \). For this reason, it is desirable to examine the results of applying MV AB to data that have been subjected to various amounts of low-pass filtering. Such filtering can be performed in different ways and at various levels of sophistication but for present purposes the simplest approach suffices. It consists of calculating a set of consecutive non-overlapping means, each over \( Q \) high-resolution data points say, thus generating a new smoothed data set containing \( \tilde{M} = M/Q \) (or, more precisely, the integer part of \( M/Q \)) points. MV AB is then performed for each choice of \( Q \) and the results for \( \hat{n} \) are compared.

Figure 8.8 shows the results of such a study for the AMPTE/IRM event in Figure 8.2. The calculations are based on a data interval containing 2048 magnetic field vectors measured at 32 samples per second and centred at 05:18:53:03 UT. The \( Q \) values used are \( Q = 2^n, n = 0, 1, 2 \ldots 9 \), for which the corresponding \( \tilde{M} \) values are \( \tilde{M} = 2^m, m = 11, 10, 9 \ldots 2 \). The figure shows the resulting ten normal vectors as dots on a nominal magnetopause tangent plane, defined by the vectors \( x_1 \) and \( x_2 \) in figure 8.2. Thus the reference normal vector in Figure 8.8, based on the usual 4.35 s averages (\( Q \approx 140 \)) is the eleventh dot, located at the origin and surrounded by its error ellipse (shown as a dashed curve). It is seen that the normals for \( Q \leq 32 \) are located in a tight cluster which is not centred at the origin but is displaced by approximately one degree from it. For \( Q > 32 \) the normal directions start to scatter increasingly widely and without an apparent pattern. All of the normals are consistent with the reference normal at the origin and its surrounding error ellipse. However, the reverse statement does not hold: the reference normal at the origin falls significantly outside the error cones associated with the smallest \( Q \) values (ellipse for \( Q = 1 \) is shown in the figure). We note that the tight clustering of the normal vectors for \( Q \leq 32 \) indicates that for instrumental and other reasons little noise is present in the data at the corresponding frequencies. It follows that for this event the \( M^{-1/2} \) dependence of the error estimates is inappropriate for the smallest \( Q \) values, i.e., for the largest \( M \) values: it leads to analytical as well as bootstrap errors that are unrealistically small. For \( Q = 32 \) the error ellipse has grown to include the point at the origin while remaining significantly smaller than the (dashed) ellipse of the reference normal. Nevertheless, because of the possible bias mentioned above it remains unclear whether the normal for \( Q = 32 \) and its ellipse represent better estimates than the reference normal and its ellipse.

Experiments of the type described above are useful as a way to establish how robust
Figure 8.8: Influence of different averaging intervals, $Q$, on normal vectors obtained from MVAB: $Q$ is the number of high-resolution vector samples (32 samples/s) averaged to obtain one of the $B$ vectors used in the minimum variance analysis ($Q \approx 140$ corresponds to the standard averaging over 4.35 s used in Figure 8.7 and in all previous figures). Normal vectors for different $Q$ values are shown as dots in the $x_2x_1$ plane with $x_2$ and $x_1$ from Figure 8.2 and with $x_3$ from that figure represented by the dot at the origin ($Q = 140$). Axes indicate angular deviations, $\Delta \psi_{32}$ and $\Delta \psi_{31}$ (radians), of a normal toward or away from $x_2$ and $x_1$, respectively. Analytically derived error cones, shown as elliptical projections in the $x_2x_1$ plane, are given for $Q = 1, 32$ and 140. Average of all normals (except $Q = 140$) is shown as a star surrounded by its ellipse of uncertainty.

is the normal vector estimate obtained from MVAB. Indeed, a possible “best” normal and error cone may be obtained by applying the analysis of Siscoe et al. [1968] to all of the ten normal vectors obtained in the experiment. The result is shown by the star-shaped dot in Figure 8.8 and the ellipse centred at it. The error cone obtained in this manner provides an alternate result which can be used instead of, or in addition to, the analytical error estimates from Section 8.3.1 or the bootstrap errors from Section 8.3.2. From what has been said, it is clear that the latter two estimates must be applied to high-resolution data with extreme caution. It may also be desirable to extend the experimentation to include the use of nested data segments, perhaps with various choices for the nest centre and including the possibility of excising data segments taken within a current layer where there is evidence
that a substructure was sampled that had orientation different from that of the main layer. The hodogram pair provides an indispensable tool for the identification of such intervals and also for the choice of optimal beginning and end points of the total data interval used for MVAB.

8.4 Other Applications

The variance analysis technique developed in the previous sections also has applications to fields other than the magnetic field. In the context of finding the orientation of a quasi-one-dimensional layer, the important requirement that must be satisfied is that either the normal component or the two tangential components of the field analysed remain constant across the layer. Two examples are given below, along with a brief discussion of MVAB as it applies to two-dimensional structures, such as surface waves on a current sheet or magnetic flux ropes.

8.4.1 Minimum Variance Analysis on Mass Flux (MVA ρv)

The mass flux across a layer such as a shock or a rotational discontinuity should be independent of the coordinate z perpendicular to the layer provided the structure of the layer is time independent. In a frame moving with the layer, we then have

\[ \nabla \cdot \rho v' = 0 \]  

(8.31)

which, assuming as before that \( \partial / \partial x = 0 \) and \( \partial / \partial y = 0 \), yields

\[ \frac{\partial}{\partial z} \rho v'_z = 0 \]

(8.32)

Here \( \rho \) is the mass density and \( v' \) is the velocity vector in a frame moving with the layer, i.e., for each measured velocity, \( v^{(m)} \), in the spacecraft frame we have

\[ v^{(m)'} = v^{(m)} - u_n \hat{n} \]

(8.33)

where \( u_n \hat{n} \) is the unknown velocity of the moving frame. It is important to note that current layers and other discontinuities in space are almost always in a state of motion: they are observed as they move past an observing spacecraft at speeds that usually greatly exceed the spacecraft velocity. Because there is a net mass flow across a rotational discontinuity (RD) or a shock we have \( \langle v \rangle \cdot \hat{n} \neq u_n \) so that \( u_n \) and \( \hat{n} \) are both unknown; only for a tangential discontinuity (TD) do we have \( u_n = \langle v \rangle \cdot \hat{n} \). An additional complication that we do not address here is that the current sheet may be in a state of acceleration or deceleration so that \( u_n \) is not constant during the crossing.

Because of the fact that the unknown normal vector, \( \hat{n} \), enters in the expression for the vector field \( \rho v' \) from which the variance matrix, \( M^{\rho v} \), is constructed, one must now use MVA in an iterative fashion. For example, one may first put \( u_n = 0 \) in order to obtain an initial \( \hat{n} = x_3 \) vector from MVA applied to the measured set \( \{ \rho^{(m)}v^{(m)} \} \); this \( \hat{n} \) vector is then used with a chosen value for \( u_n \) (for RDs and shocks) or with \( u_n = \langle v \rangle \cdot \hat{n} \) (for TDs) in order to produce a new set \( \{ \rho^{(m)}v^{(m)'} \} \). MVA applied to this set will then yield a new \( \hat{n} \) vector to replace the initial one. The calculation is then repeated until \( \hat{n} \) no longer changes:
8.4. Other Applications

usually only a few iterations are needed but for large values of $|u_n|$ convergence problems may arise. In that case, a stepwise approach can be adopted in which one first iterates to obtain the $\hat{n}$ vector corresponding to a smaller $|u_n|$ value and then uses that vector as the initial guess for a larger $|u_n|$, and so on. Rapidly converging iterative schemes for finding the smallest or largest eigenvalue of a matrix also exist and can be adapted to the present situation. Since the problem is nonlinear, difficulties with uniqueness may arise. For example, two iterations using the same value of $u_n$ but starting from $x_3$ and $-x_3$, respectively, generally do not converge to the same final $\hat{n}$ vector. However, it is usually easy to decide which answer makes physical sense.

For a TD, normal vectors for different $u_n$ values are calculated in the fashion described above until the $u_n$ value has been found for which the condition $u_n = \langle v \rangle \cdot \hat{n}$ is satisfied. Making the proper choice of $u_n$ for RDs and shocks is a non-trivial task which requires observational information beyond $\rho^{(m)}$ and $v^{(m)}$. If a good minimum variance direction has been obtained from MVAB, one may determine $u_n$ by maximising error cone overlap for the MVAB normal and the MVAfv normal and then choosing the normal at the centre of the overlap region. If the MVAB normal is poorly determined but the maximum variance eigenvector, $x_1$, from MVAB has small errors, one may determine $u_n$ from the condition $\hat{n} \cdot x_1 = 0$. Other possibilities also exist. For RDs $u_n$ could be chosen such that $\langle v' \rangle \cdot \hat{n} = \pm \langle v_A \rangle \cdot \hat{n}$, $v_A$ being the measured Alfven velocities; for shocks one could use the Rankine-Hugoniot conditions, as discussed in detail in Chapter 10. Alternatively, timing differences from dual or multiple spacecraft measurements can be used to obtain $u_n$. Still another possibility (incompletely explored to date) is that one or more of the eigenvalues, $\lambda_1$, $\lambda_2$, $\lambda_3$, may exhibit an extremum at the correct value of $u_n$.

Provided $\lambda_2 \gg \lambda_3$, the process described above will provide a normal vector derived from plasma data alone but the direction of this vector depends somewhat on the choice of $u_n$, the magnitude of which remains unknown (unless there is other evidence to specify it as discussed above). Note that the sign of $u_n$ is usually known. For example, a magnetopause crossing from the magnetosphere to the magnetosheath, such as the one in Figure 8.2, must have $u_n < 0$.

As an illustration, the minimum variance analysis on $\rho v$ described in this section has been applied to the plasma data from the AMPTE/IRM magnetopause crossing in Figure 8.2. The results, for current-layer normal velocity $u_n = 0$ and for $u_n = -5$ km/s, are given in Table 8.2; hodograms for $u_n = -5$ km/s are shown in Figure 8.9. In this figure, the mass fluxes have been converted to weighted velocities, $\rho v'/\langle \rho \rangle$ where $\langle \rho \rangle = 19.4$ particle masses per cm$^3$. Information about plasma composition is required in order to determine the mass density but was not obtained by the AMPTE/IRM plasma instrument. If the composition ratios remain the same throughout the layer, then the normal-vector determination from $\rho^{(m)}v^{(m)}$ does not depend on the actual values of those ratios. However, gradients in composition, which might be present across the magnetopause, would corrupt the determination.

From the results in Table 8.2 one can calculate that, for $u_n = 0$, the $x_3$ vector derived from MVAfv forms an angle of 5.9° with the minimum variance direction for $B$ given in Figure 8.2 while for $u_n = -5$ km/s the angle has increased to 7.2°. Thus an attempt to minimise the angle would lead to $u_n > 0$; this would be inconsistent with the fact that the magnetopause was traversed in the direction from the magnetosphere to the magnetosheath which requires $u_n < 0$. This is also the case if the TD condition $u_n = \langle v \rangle \cdot \hat{n}$, which should be applicable to this event, is implemented: it leads to $\hat{n} = (0.8584, -0.5096, -0.0577)$.
Figure 8.9: Hodograms and minimum variance results for the weighted velocity, $\rho v'/\langle \rho \rangle$ (km/s), for event in Figure 8.2 with $u_n = -5$ km/s. The average density $\langle \rho \rangle$ corresponds to 19.4 particles/cm$^3$; the range is 9.2–26.2 particles/cm$^3$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\lambda_i$(km/s)$^2$</th>
<th>$x_i$</th>
<th>$\langle \rho v' \rangle x_i / \langle \rho \rangle$(km/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7044.6</td>
<td>(-0.4388, -0.8410, 0.3164)</td>
<td>245.80</td>
</tr>
<tr>
<td>2</td>
<td>832.1</td>
<td>(-0.2453, -0.2265, -0.9426)</td>
<td>2.75</td>
</tr>
<tr>
<td>3</td>
<td>162.9</td>
<td>(0.8644, -0.4913, -0.1071)</td>
<td>2.1</td>
</tr>
</tbody>
</table>

$u_n \simeq + 6$ km/s and to a decrease of the angle between the MVAB and the MVA$\rho v$ normals to 4.5°. Further discussion of the inconsistent sign for $u_n$ is given in Section 8.5.

Because it is more difficult to accurately measure $\rho$ and $v$ of the plasma (see Chapter 6) than to measure $B$, the measurement errors in $v$ and, in particular, in $\rho$ are at present sufficiently large so as to limit the ability of the analysis techniques described above to produce an accurate normal vector based on $\rho v$ or a reliable value of $u_n$. It is emphasised that by requiring MVAB and MVA$\rho v$ error-cone intersection to occur in order for $u_n$ to be determined and by placing the final choice in the overlap region, this type of analysis will lead to rejection of poor cases and will place more emphasis on the normal vector determination having the smaller error cone. In our example, this means placing more emphasis on $B$ data and less on $\rho v$ data. A more sophisticated technique which incorporates several conservation laws in one grand optimisation has been proposed recently [Kawano and Higuchi, 1996] but remains untested. Such approaches hold considerable promise but they
may also encounter difficulties related to the limited quality of the plasma information. It is important that any such procedure should allow greater weight to be placed on quantities that are well determined with less weight on quantities that are less well determined.

Methods that utilise data on the two sides of the current layer but not in its interior are discussed in Chapter 10.

8.4.2 Maximum Variance Analysis on Electric Field (MVAE)

Faraday’s law requires that the two electric field components tangential to a one-dimensional time independent layer remain constant throughout the layer and on its two sides. It is important to note that the electric field is to be measured in a frame of reference moving with the layer. There are certain classes of current layer for which this constancy of the tangential electric field can be used for determination of a vector normal to the layer. What is required is that the electric field component normal to the layer undergoes a large change as it often does at the magnetopause where the tangential flow and/or the tangential $B$ field components usually change direction and/or magnitude by substantial amounts across the layer. In that case, $\lambda_1$ is large and the maximum variance direction will serve as a good predictor of the normal to the layer. The variance ellipsoid would ideally be cigar shaped with $\lambda_1 \gg \lambda_2 \simeq \lambda_3$ (in practice one is more likely to find $\lambda_2 > \lambda_3$ or even $\lambda_2 \gg \lambda_3$ rather than $\lambda_2 \simeq \lambda_3$).

The electric field observed in a frame moving with the layer is

$$E^{(m)'} = E^{(m)} + u_n \hat{n} \times B^{(m)}$$

where $E^{(m)}$ is the field measured in the spacecraft frame. The maximum variance eigenvector, $x_1$, of the matrix

$$M_\mu = \langle E'_\mu E'_\nu \rangle - \langle E'_\mu \rangle \langle E'_\nu \rangle$$

then becomes our predictor of the normal $\hat{n}$. It is noted that, as in Section 8.4.1, the resulting normal vector depends on the choice of $u_n$, which may itself be a function of time, and that the calculation must be done iteratively since $E'$ itself contains the unknown normal $\hat{n}$.

To date, this type of maximum variance analysis has not been performed on actual measured electric field vectors $E^{(m)}$. Rather, the convection electric field $E_c^{(m)} = -v^{(m)} \times B^{(m)}$, calculated from measured plasma velocities and magnetic fields, has been used as a proxy for $E^{(m)'}$. (A variant of MVAE that remains untested to date is to use $-v_e^{(m)} \times$
The comments, given in Section 8.4.1, concerning the determination of $u_n$, apply to MVAE as well.

We point out that, even when the correct value of $u_n$ is known, the normal vectors obtained from MVAE are (usually weakly) frame dependent: if MVAE is performed for the same event in two frames that move relative to each other along the current sheet, the resulting normal vectors are not identical. This effect, which is caused by the noise, may become large if one moves to a frame of reference in which the noise in $E'$ dominates the convective field, $-v \times B$. 

Figure 8.10: Hodograms and maximum variance results for $E^{(m)}_c = -v^{(m)} \times B^{(m)}$ (mV/m) for the event in Figure 8.2 with $u_n = -5$ km/s.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\lambda_i$ (mV/m)$^2$</th>
<th>$\xi_i$</th>
<th>$\langle E \rangle \cdot \xi_i$ (mV/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17.05</td>
<td>(0.9017, 0.4309, 0.0361)</td>
<td>7.55</td>
</tr>
<tr>
<td>2</td>
<td>0.50</td>
<td>(0.0627, 0.0476, 0.9969)</td>
<td>-0.13</td>
</tr>
<tr>
<td>3</td>
<td>0.21</td>
<td>(0.4278, 0.9011, 0.0699)</td>
<td>-0.03</td>
</tr>
</tbody>
</table>
Table 8.3: Predictions from MVAE for the event in Figure 8.2.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>$u_n = 0$ km/s</th>
<th>$u_n = -5$ km/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>(0.8911, -0.4533, 0.0197)</td>
<td>(0.9017, -0.4309, 0.0361)</td>
</tr>
<tr>
<td>$\Delta \varphi_{31}$</td>
<td>0.0289 rad</td>
<td>0.0290 rad</td>
</tr>
<tr>
<td>$\Delta \varphi_{32}$</td>
<td>0.2943 rad</td>
<td>0.2913 rad</td>
</tr>
<tr>
<td>$\Delta \varphi_{21}^*$</td>
<td>0.0451 rad</td>
<td>0.0453 rad</td>
</tr>
<tr>
<td>$\langle E \rangle \cdot x_2 \pm \Delta \langle E \rangle \cdot x_2$</td>
<td>$-0.25 \pm 0.38$ mV/m</td>
<td>$-0.13 \pm 0.38$ mV/m</td>
</tr>
<tr>
<td>$\langle E \rangle \cdot x_3 \pm \Delta \langle E \rangle \cdot x_3$</td>
<td>$0.02 \pm 0.25$ mV/m</td>
<td>$-0.03 \pm 0.25$ mV/m</td>
</tr>
</tbody>
</table>

*From equation 8.23 with $\lambda_3$ replaced by $\lambda_2$.

As an illustration of MVAE, we have applied it to the AMPTE/IRM magnetopause crossing in Figure 8.2 with the results shown in Figure 8.10 and in Table 8.3. Note that $x_1$ from MVAE with $u_n = 0$ forms an angle of 2.9° with $x_3$ from MVAB in Figure 8.2 while for $u_n = -5$ km/s that angle has increased to 4.4°. Therefore an attempt to minimise the angle between the maximum variance eigenvector from $E$ and the minimum variance eigenvector from $B$ (Figure 8.2) would again lead to $u_n > 0$, in conflict with the actual direction of traversal of the magnetopause which requires $u_n < 0$. The TD condition, $u_n = \langle v \rangle \cdot \mathbf{n}$, leads to $u_n \approx -48$ km/s but the resulting normal vector $(0.9598, -0.2205, 0.1736)$ is not believable: it forms an angle of 19.0° with $x_3$ from MVAB.

An alternate approach to MVAE, in which a suitably defined residue of the integrated Faraday equation is minimised to yield both $\mathbf{n}$ and $u_n$, has been described recently by Terasawa et al. [1996]. The method is based on the constancy of $B^{(m)} \cdot \mathbf{n}$ as well as of $E^{(m)'} \times \mathbf{n}$; minimisation of the variance of the tangential electric field plays a key role in the determination of $u_n$. As stressed in Section 8.4.1, caution should be exercised in accepting the results from such combined calculations because it is not clear that they properly weight information of high and less high quality. We have applied the method of Terasawa et al. to the AMPTE/IRM crossing in Figure 8.2 with the result $\mathbf{n}_T = (0.8945, -0.4465, 0.0247)$ and $u_n = -1.5$ km/s. We also find $\langle v \rangle \cdot \mathbf{n}_T = -5$ km/s and $\langle B \rangle \cdot \mathbf{n}_T = -0.24$ nT. These results are reasonable: $\langle B \rangle \cdot \mathbf{n}_T$ is near zero and both $u_n$ and $\langle v \rangle \cdot \mathbf{n}_T$ are negative, as required for an earthward moving TD. An inconsistency is that $u_n \neq \langle v \rangle \cdot \mathbf{n}_T$ but this discrepancy is well within uncertainties. However, we find that $\mathbf{n}_T$ is close to the $\mathbf{n}$ vector from MVAE with $u_n = 0$ (see Table 8.3): the two vectors form an angle of only 0.4° whereas the angle between $\mathbf{n}_T$ and the MVAB vector $x_3$ in Figure 8.2 is 3.3°. In this example, the method evidently puts most of the weight on the constancy of the tangential electric field, $E^{(m)'} \times \mathbf{n}_T$, and relatively little weight on the constancy of the normal magnetic field, $B^{(m)} \cdot \mathbf{n}_T$.

A number of items relative to MVAE and to the method of Terasawa et al. remain unexplored or incompletely understood at present. Included are the details of the inter-relationship between the two methods and their mutual relationship to the existence and quality of a so-called deHoffmann-Teller frame (see Chapter 9), i.e., a moving frame of reference in which the electric field is absent or small.
8.4.3 Application to 2-D Structures

We now discuss briefly a quite different use of MVAB, introduced by Walthour and Sonnerup [1995]. They applied this technique to magnetic field data obtained from remote sensing of two-dimensional bulges, such as flux transfer events or surface waves on the magnetopause, moving past an observing spacecraft. During the event, the spacecraft is assumed to be located entirely on one side of the current layer; it cannot penetrate the bulge or the layer. The details of this application are too lengthy to be presented here but some of the results are of interest. Linear ideal MHD theory can be used to show that the smallest eigenvalue of $M_{\mu\nu}^B$ (defined by equation 8.8) should be zero, i.e., $\lambda_3 = 0$, and that the corresponding eigenvector $x_3$ should be tangential to the unperturbed layer, forming a certain angle, $\theta_N$, with the invariant direction (the axis) of the bulge. The angle $\theta_N$ is obtainable from the direction of motion of the bulge relative to the unperturbed field and the so-called stretching factor, $R$, which enters into the MHD perturbation analysis of flow over small-amplitude bulges [Sonnerup et al., 1992]. A curl-free magnetic field corresponds to $R = 1$ and $\theta_N = 0$; in that special case the field component along the invariant direction is exactly constant so that $x_3$ is along the bulge axis. In the general case, it can also be shown [Khrabrov and Sonnerup, 1998b] that either the intermediate variance eigenvector, $x_2$, or the maximum variance eigenvector, $x_1$, should be normal to the unperturbed layer and that the ratio of maximum to intermediate eigenvalue $\lambda_1/\lambda_2 = R^2/\cos^2 \theta_N$ in the former case and $\lambda_2/\lambda_1 = R^2/\cos^2 \theta_N$ in the latter case. Several actual magnetopause events have been analysed for which these predicted properties of $M_{\mu\nu}^B$ were verified.

Another illustration is the case of a spacecraft traversing a force-free flux rope in which the axial current, along the $z$ axis say, is uniformly distributed over the circular cross section of the rope. It is simple to show that magnetic data taken along a straight-line trajectory through the interior of the rope will produce $\lambda_3 = 0$ and a minimum variance direction that is not along the rope axis but is instead perpendicular to it and is, in fact, along the projection of the trajectory onto the perpendicular $xy$ plane. Application of MVAB to flux ropes is discussed further by Elphic and Russell [1983] and Lepping et al. [1990].

From these examples the following lesson can be drawn. A nearly one-dimensional current layer will produce $\lambda_3 \ll \lambda_1$ but the converse is not necessarily true: if one finds $\lambda_3 \ll \lambda_1$ for a measured data set, it does not automatically follow that a 1-D current layer has been traversed. Furthermore, if one seeks an interpretation in terms of a 2-D structure instead, one cannot necessarily conclude that the minimum variance direction coincides with the axis of the structure.

8.5 Discussion of AMPTE/IRM Event

In this chapter we have used one particular magnetopause crossing by the AMPTE/IRM spacecraft in order to illustrate various applications of the minimum/maximum variance analysis and associated error estimates. For benchmarking purposes, we provide, in the Appendix, the 16 samples of field $B$, number density $N^*$, and velocity $v$ for this event that were used in most of the calculations. We now compare the various results for the normal vector estimate, the normal field and flow component estimates, and the error estimates
for this event.

An overview of the normal vector orientations is given in Figure 8.11 where each normal vector is shown as a dot in the \((x_1, x_2)\) plane. The axes of this plane are provided by the maximum and intermediate eigenvectors in Figure 8.2 and the distances along the \(x_1\) and \(x_2\) axes represent angular deviations, \(\Delta \varphi_3\) and \(\Delta \varphi_2\), in radians. Thus the normal vector \(x_3\) from Figure 8.2 appears as point 1 at the origin with its error-cone cross section (from equation 8.23) being the dashed ellipse with major axis along \(x_2\) and minor axis along \(x_1\). The minimum variance calculation with constraint \((B) \cdot \hat{n} = 0\) gives point 2 and the normal vector of Siscoe et al. (from the expression 8.17) gives point 3 (no error cones shown). The average bootstrap estimate of \(\hat{n}\), calculated as the eigenvector corresponding to the maximum \(\lambda\) value in the analysis of Siscoe et al., applied to the bootstrap normals, is point 4 which is very nearly coincident with point 1. It is surrounded by its slightly tilted error-cone cross section, the major and minor axes of which are along the eigenvectors corresponding to intermediate and minimum \(\lambda\) values in the analysis of Siscoe et al. and the semiaxis lengths of which are the square root of those \(\lambda\) values. This ellipse is nearly the same as the dashed ellipse for point 1. Point 5, surrounded by its error-cone cross section (or, more precisely, the projection of that cross section onto the \(x_1,x_2\) plane), is similarly the normal vector obtained by applying the analysis of Siscoe et al. to the \(K = 9\) normal vectors on the plateau in Figure 8.8. The ellipse is small and falls inside the dashed error ellipse associated with point 1, suggesting that the latter error estimate is reasonable. The clustering of all of points 1–5 within the dashed ellipse reinforces the view that the error estimate from 8.23 is valid.

Points 6, 7 and 8, which are located along a straight line, are normal vectors from MVAE with \(u_n = 0 \text{ km/s}, u_\| = +12 \text{ km/s}\) (point of maximum error-cone overlap with MVAB) and \(u_n = +20 \text{ km/s}\) (from condition \(\hat{n} \cdot x_1 = 0\) with \(x_1\) from MVAB), respectively. The elliptical error-cone cross section refers to point 6. For MVAE, the TD condition \(u_n = \langle v \rangle \cdot \hat{n}\) yields \(u_n = -48 \text{ km/s};\) the corresponding normal vector is located outside of the diagram at \(x_1 = -0.13, x_2 = -0.30\). Points 9, 10 and 11, which are also located along a straight line, are normal vectors from MVAR\(\nu\) with \(u_n = 0 \text{ km/s}, u_\| = +6 \text{ km/s}\) (from the TD condition, \(u_n = \langle v \rangle \cdot \hat{n}\) and \(u_\| = +15 \text{ km/s}\) (maximum error-cone overlap and also \(\hat{n} \cdot x_1 \approx 0\)). The large elliptical error-cone cross section refers to point 10. Finally, point 12 is the normal from Terasawa’s method. The relations \(\langle B \rangle \cdot \hat{n} = 0\) and \(\langle v \rangle \cdot \hat{n} = 0\), calculated from the 16 data points in the appendix, are also shown in Figure 8.11 as lines B–B and V–V, respectively. To the right of B–B \(\langle B \rangle \cdot \hat{n} < 0\) and to the left of it \(\langle B \rangle \cdot \hat{n} > 0\). Similarly, \(\langle v \rangle \cdot \hat{n} > 0\) above V–V and \(\langle v \rangle \cdot \hat{n} < 0\) below it.

Point 6 \((u_n = 0 \text{ km/s}, \langle v \rangle \cdot \hat{n} = -3.6 \text{ km/s})\) represents the best determination of \(\hat{n}\) from MVAE that is approximately consistent with \(u_n \leq 0\) and with the structure being that of a TD. The result has difficulties: (i) the true value of \(u_n\) must in fact be negative and must coincide with \(\langle v \rangle \cdot \hat{n}\); (ii) the error cone for point 6 does not contain points 1–5 and has only a small overlap with the error cone surrounding point 1. To bring about maximal error-cone overlap \(u_n = +12 \text{ km/s}\) is needed. Similarly, point 10 \((u_n = \langle v \rangle \cdot \hat{n} = +6 \text{ km/s})\) represents the best determination of \(\hat{n}\) and \(u_\|\) from MVAR\(\nu\) for this TD. Again \(u_\|\) has the wrong sign. The error cone for this point is large enough to include points 1–5 but optimal overlap would require \(u_n \approx +15 \text{ km/s}\). These results all suggest a systematic deviation of the normals based on plasma measurements from the more reliable normals obtained from the magnetic data alone: they indicate that the plasma velocity component along \(\hat{n}\) may be too large. If that component is decreased by 12 km/s, say, point 6 from MVAE
Figure 8.11: Overview of normal-vector determinations and errors for event in Figure 8.2. The $x_2x_1$ plane is shown with $x_1$ and $x_2$ given in Figure 8.2. The reference normal vector $x_3$ from Figure 8.2 is the point at the origin; deviations $\Delta \varphi_{31}$ and $\Delta \varphi_{32}$ (radians) are shown as distances along $x_1$ and $x_2$, respectively. The normal vectors, numbered from 1 to 12 are described in Section 8.5. The line B–B corresponds to $\langle B \rangle \cdot \hat{n} = 0$; on the left of that line $\langle B \rangle \cdot \hat{n} > 0$ and on the right $\langle B \rangle \cdot \hat{n} < 0$. Similarly, the line V–V corresponds to $\langle v \rangle \cdot \hat{n} = 0$; above the line $\langle v \rangle \cdot \hat{n} > 0$ and below it $\langle v \rangle \cdot \hat{n} < 0$. Both B–B and V–V are calculated from the data set in the Appendix. Ellipses represent error-cone cross sections, at unit distances from the cone vertex, projected onto the $x_2x_1$ plane.
would move to the location of point 7 where maximum error-cone overlap with the MVAB normal (point 1) occurs. Similarly, the TD point 10 from MVAB would move to the vicinity of point 11, again providing maximum error-cone overlap with the MVAB normal. However, the actual values of \( u_n \) for the TD points at their new locations would remain at 0 km/s and +6 km/s, respectively which is inconsistent with the requirement \( u_n < 0 \).

Another indication of a systematic error in the plasma velocities is that the MVAB normal vector (point 1) has \( \langle v \rangle \cdot \hat{n} = \langle v \rangle \cdot x_3 = +8 \text{ km/s} \), i.e., it is well above the line \( V-V \) at which \( \langle v \rangle \cdot \hat{n} = 0 \). This result suggests that the measured normal velocity component of the plasma is too large by at least 8 km/s and perhaps more, depending on how rapid was the actual inward motion of the magnetopause past the spacecraft, i.e., depending on how negative \( u_n \) was. In Chapter 9, the magnetopause velocity \( u_n \) is calculated as the component of the deHoffmann-Teller frame velocity \( V_{HT} \) along \( x_3 \), the MVAB reference normal. When the data set in the Appendix is used to obtain \( V_{HT} \), the result is either \( u_n = V_{HT} \cdot x_3 = (5.9 \pm 3.9) \text{ km/s} \) or \( u_n = (8.1 \pm 4.2) \text{ km/s} \), depending on certain details of the calculation. These results again suggest the presence of a small systematic error (although, as noted in Chapter 9, somewhat longer data segments for the calculation of \( V_{HT} \) do in fact yield the required negative values of \( u_n \)).

Because of the likely presence in the plasma velocity data of such a systematic error, perhaps caused by the plasma instrument being in its half-sweep mode as discussed below, it does not appear possible to determine the actual negative value of \( u_n \) for this event. However, it seems to have been small, probably in the range \( 0 > u_n > -10 \text{ km/s} \).

The reason for the discrepancy between the normals based purely on \( B \) (points 1–5 in Figure 8.11) and those based partially or wholly on the plasma measurements (points 6 and 10, say, in Figure 8.11) is not understood. But it may be related, at least in part, to the fact that the plasma analyser was in its so-called half-sweep mode during the crossing. In this mode the ion energy range sampled is reduced from \( 20 \text{ eV/q} < E < 40 \text{ keV/q} \) to \( 150 \text{ eV/q} < E < 5.3 \text{ keV/q} \). Simulations of the instrument under typical magnetopause conditions indicate that systematic directional errors of the plasma flow velocity of a few degrees may occur in the half-sweep mode, along with an overestimate of the velocity magnitude and an underestimate of the density [G. Paschmann, private communication, 1996]. Some time aliasing is also likely to be present. Keeping these effects in mind, we conclude that the approximate agreement, illustrated in Figure 8.11, between the plasma-based normal vectors from MVAB and MVAE and the various magnetically-based normal vectors from MVAB is rather remarkable.

8.6 Summary

We summarise the material presented in this chapter as follows.

1. The main application of minimum variance analysis (MVA) is to the task of estimating \( \hat{n} \), the direction perpendicular to approximately one-dimensional structures, such as current layers and plane wave fronts, from a set of magnetic field data measured by a single spacecraft during traversal of the structure.

2. MVA applied to magnetic field data (MVAB) consists of constructing the variance matrix (equation 8.8) and then finding its eigenvalues \( (\lambda_1, \lambda_2, \lambda_3) \) and corresponding eigenvectors \( (x_1, x_2, x_3) \). The direction of minimum variance, \( x_3 \), is used as an
estimator of $\hat{n}$, the corresponding eigenvalue, $\lambda_3$, being the variance of the field component along $x_3$. A small value of $\lambda_3$, compared to $\lambda_2$ and $\lambda_1$, generally signals a good determination of $\hat{n}$. Extreme caution must be exercised in cases where the separation between the minimum eigenvalue, $\lambda_3$, and the intermediate eigenvalue, $\lambda_2$, is small: the matrix is then nearly degenerate and small changes in the data interval used for the analysis may lead to an interchange of the eigenvectors $x_3$ and $x_2$ (Section 8.2.5). The normal vector obtained is unreliable in such cases. A simple rule of thumb is $\lambda_2/\lambda_3 \geq 10$ for a good application of MVA to a relatively small data set ($M < 50$, say).

3. Use of the Siscoe et al. formulation (equation 8.17) in place of 8.8 for determination of $\hat{n}$ is not recommended, except perhaps for tangential discontinuities when $\lambda_2/\lambda_3$ is not large; however, for such cases MVA with constraint $\langle B_3 \rangle = 0$ (Section 8.2.6) gives similar results. Nevertheless, the Siscoe et al. analysis has been found useful for characterising a cluster of normal vectors obtained from MVA bootstrap calculations or from analysis of nested data segments.

4. We recommend (Section 8.3.1) that statistical errors be estimated by use of equations 8.23 and 8.24 and that a number of earlier estimates that have appeared in the literature be discarded. For small errors, the formula 8.23 gives results that nearly duplicate those obtained from the bootstrap technique (Section 8.3.2). In that case, the bootstrap average normal is also essentially the same as the minimum variance normal obtained by a single application of MVA to the measured data set. Thus there is no clear reason to perform the more time consuming bootstrap calculations to obtain $\hat{n}$ and its angular uncertainties. However, in cases where the signal-to-noise ratio is small ($\lambda_3$ is comparable to $\lambda_2$ and perhaps $\lambda_1$) and the number of data points, $M$, used is not correspondingly large or when systematic errors are present, the bootstrap calculation may possibly give more realistic error estimates than equation 8.23. There are modest discrepancies between the analytical error estimate 8.24 for the average normal-field component and the corresponding bootstrap result, calculated as described by Kawano and Higuchi [1995]. The reasons for these deviations are not clearly understood but, at present, our recommendation is to use the results from equation 8.24. It is also noted that the estimates given by equations 8.23 and 8.24 are inversely proportional to $M^{1/2}$ so that very small errors may result when large data sets are used. Although this behaviour is expected for random noise, it may lead to the underestimation of uncertainties when data at high time resolution are used. It must also be remembered that in many applications additional, systematic errors, not described by equations 8.23 and 8.24, may be present as well.

5. It is recommended that MVA be performed on nested sets of data intervals, centred at or near the middle of the structure being measured. Time stationarity can be checked in this manner (Section 8.3.4) but care must be taken to avoid including adjoining structures or turbulence that may degrade the quality of the normal vector determination. Similarly, it may sometimes be important to perform low-pass filtering of the data before MVA is applied (Section 8.3.5) in order to remove higher frequency wave activity in the interior of the layer. Such waves may have propagation vectors very different from $\hat{n}$ and therefore may degrade the determination of
8.6. Summary

In most cases averaging the data over consecutive non-overlapping time intervals will provide filtering of sufficient quality.

6. We have pointed out (Section 8.4.3) that certain two-dimensional structures will lead to a magnetic variance matrix having $\lambda_3$ equal to zero, ideally, and in reality having $\lambda_3 \ll \lambda_2, \lambda_1$. It must be remembered, therefore, that the existence of a direction of low variance for a data set does not guarantee that the structure sampled by the spacecraft was a one-dimensional current sheet or wave.

7. There are also relevant applications of MVA to vector fields other than $B$. In particular, mass flow conservation (MVA$_{\rho v}$; see Section 8.4.1) across a one-dimensional layer implies that the minimum variance direction for $\rho v^\prime$, where $v^\prime = v - u_n \hat{n}$ is the plasma velocity in a frame moving with the layer, should be a good normal direction. A difficulty with this application is that the value of the speed, $u_n$, of the layer must be obtained separately from other measurements or conditions which is a simple task only for tangential discontinuities. Similarly, the maximum variance direction for the convection electric field $E^\prime = -(v - u_n \hat{n}) \times B$ in the frame moving with the layer should be normal to the layer as a consequence of Faraday’s law (MVAE; see Section 8.4.2). Again, the value of $u_n$ must be obtained separately. Experience indicates that this maximum variance analysis on $E^\prime$ sometimes gives a reasonably reliable normal vector, the requirement being that $\lambda_1 \gg \lambda_2, \lambda_3$, even where minimum variance analysis on $B$ does not. The error estimate 8.23 for $|\Delta \varphi|_{21}$ should be modified by replacing $\lambda_3$ by $\lambda_2$ in the application to the MVAE normal.

8. Recently, an analysis technique has been proposed in which the constancy of the normal component of $B$ and of the tangential components of $E$ are both satisfied in a single optimisation that yields $\hat{n}$ as well as $u_n$ [Terasawa et al., 1996]; an analytical solution to this optimisation problem and associated error estimates have been developed by Khrabrov and Sonnerup [1998c]. Although the utility of the method by Terasawa et al. has not yet been widely tested, it performed well in our AMPTE/IRM sample event. However, concerns exist that it does not always provide the proper relative weighting of quantities that are measured or calculated more accurately and less accurately. This concern has been addressed to some extent in a recent generalisation of MVA to include a variety of additional conservation laws [Kawano and Higuchi, 1996]. However, this method remains untested. At present, we are therefore in favour of separate use of MVAB and MVAE (plus perhaps MVA$_{\rho v}$). The former yields a single optimal normal vector and an associated error cone. The latter yields a distinct normal vector and error cone for each chosen value of $u_n$. For tangential discontinuities, the proper value of $u_n$ can in principle be found from the condition $u_n = \langle v \rangle \cdot \hat{n}$ but there is no guarantee that the result is reasonable. In other cases, the optimal choice for $u_n$ should be that which leads to maximal overlap between the MVAB error cone and that of MVAE (and perhaps MVA$_{\rho v}$); the optimal $\hat{n}$ could then be taken at the centre of the overlap region. If the normal vector from MVAB has large errors, an alternate choice for $u_n$ may be such that the corresponding normal vector from MVAE or MVA$_{\rho v}$ is perpendicular to the maximum variance direction from MVAB. Or, following Terasawa et al., one may search for the $u_n$ value that gives a minimum in the sum $(\lambda_2 + \lambda_3)$, obtained from MVAE. One advantage of the separate use of MVAB, MVAE, and MVA$_{\rho v}$ is that
inconsistencies in the various determinations of \( \hat{n} \) may be revealed, as illustrated in Section 8.5. Such inconsistencies, which point to possible systematic errors in some of the measured quantities, may remain hidden if a single combined merit function is optimised.

**Appendix**

The 16 AMPTE/IRM data points, from 05:18:20–05:19:26 UT on October 19, 1984, used in Figures 8.2, 8.6, 8.9, and 8.10, are given in Table 8.4 for use in benchmark tests of MVA and other programs. It is noted that these data differ slightly from those used in Somerup et al. [1990], probably as a result of round-off errors during coordinate transformations. Therefore normal vectors given in that paper differ slightly from those reported here.

Table 8.4: The 16 AMPTE/IRM data points from 05:18:20–05:19:26 UT on October 19, 1984 used in various demonstrations in this chapter

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Units: \( B \) (nT); \( v \) (km/s); \( N^* \) (particles/cm\(^3\)); time \( T \) (sec), starting at 05:18:20.49 UT.

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Bibliography


